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Numerical Analysis (10th Edition)

Chapter 5.5, Problem 1E 1 Bookmark Show all steps: ON

Problem

Use the Runge-Kutta-Fehlberg method with tolerance $TOL = 10^{-4}$, $hmax = 0.25$, and $hmin = 0.05$ to approximate the solutions to the following initial-value problems. Compare the results to the actual values.

a. $y' = te^{3t} - 2y,\quad 0 \le t \le 1,\quad y(0) = 0$; actual solution $y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}$.

b. $y' = 1 + (t - y)2$, $2 \le t \le 3$, $y(2) = 1$; actual solution $y(t) = t + 1/(1 - t)$.

c. $y' = 1 + y/t$, $1 \le t \le 2$, $y(1) = 2$; actual solution $y(t) = t\ln t + 2t$.

d. $y' = \cos 2t + \sin 3t,\quad 0 \le t \le 1,\quad y(0) = 1$; actual solution $y(t) = \frac{1}{2}\sin 2t - \frac{1}{3}\cos 3t + \frac{4}{3}$.

Step-by-step solution

Step 1 of 6

The initial condition gives

$$t_0 = 0$$
$$w_0 = 0$$

To determine w_1 using

$$\text{Let } h = h\max = 0.25$$

Let $k_1, k_2, k_3, k_4, k_5, k_6$ be six evaluations of f

Compute $k_1, k_2, k_3, k_4, k_5, k_6$ as follows:

$$\begin{aligned} k_1 &= hf(t_0, w_0) \\ &= 0.25 f(0,0) \\ &= 0 \end{aligned}$$

$$\begin{aligned} k_2 &= hf\left(t_0 + \frac{1}{4}h, w_0 + \frac{1}{4}k_1\right) \\ &= 0.25 f\left(\frac{0.25}{4}, 0\right) \\ &= 0.25\left(0.0625e^{0.1875}\right) \\ &= 0.0188473 \end{aligned}$$

$$\begin{aligned} k_3 &= hf\left(t_0 + \frac{3}{8}h, w_0 + \frac{3}{32}k_1 + \frac{9}{32}k_2\right) \\ &= 0.25 f\left(0 + \frac{3}{8}(0.25), 0 + \frac{3}{32}(0) + \frac{9}{32}(0.01885)\right) \\ &= 0.25 f(0.09375, 0.00530156) \\ &= 0.25\left(0.09375e^{0.28125} - 0.0106\right) \\ &= 0.0284 \end{aligned}$$

$$\begin{aligned} k_4 &= hf\left(t_0 + \frac{12}{13}h, w_0 + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3\right) \\ &= 0.25 f\left(0 + \frac{12}{13}(0.25), 0 + \frac{1932}{2197}(0) - \frac{7200}{2197}(0.0188473) + \frac{7296}{2197}(0.0284)\right) \\ &= 0.25 f(0.230769, 0.032547) \\ &= 0.25\left(0.230769e^{0.692307} - 2(0.032547)\right) \\ &= 0.0990141 \end{aligned}$$

$$\begin{aligned} k_5 &= hf\left(t_0 + h, w_0 + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4\right) \\ &= 0.25 f\left(0 + 0.25, 0 + \frac{439}{216}(0) - 8(0.0188473) + \frac{3680}{513}(0.0284) - \frac{845}{4104}(0.0990141)\right) \\ &= 0.25 f(0.25, 0.032562) \\ &= 0.25\left(0.25e^{0.75} - 2(0.032562)\right) \\ &= 0.116032 \end{aligned}$$

$$\begin{aligned} k_6 &= hf\left(t_0 + \frac{h}{2}, w_0 - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5\right) \\ &= 0.25 f\left(\begin{aligned}&0 + \frac{0.25}{2}, 0 - \frac{8}{27}(0) + 2(0.0188473) - \frac{3544}{2565}(0.0284) + \frac{1859}{4104}(0.0990141) \\ &- \frac{11}{40}(0.116032)\end{aligned}\right) \\ &= 0.25 f(0.125, 0.0113969) \\ &= 0.25\left(0.125e^{0.375} - 2(0.0113969)\right) \\ &= 0.0397700 \end{aligned}$$

Compute w_1

$$\begin{aligned} w_1 &= w_0 + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6 \\ &= 0 + \frac{6}{135}(0) + \frac{6656}{12825}(0.0284) + \frac{28561}{56430}(0.0990141) - \frac{9}{50}(0.116032) \\ &\quad + \frac{2}{55}(0.0397700) \\ &= 0.0298184 \end{aligned}$$

$$\begin{aligned} \text{Let } R &= \frac{1}{h}\left(\frac{1}{360}k_1 - \frac{128}{4275}k_3 - \frac{2197}{75240}k_4 + \frac{1}{50}k_5 + \frac{2}{55}k_6\right) \\ &= \frac{1}{0.25}\left(\begin{aligned}&\frac{1}{360}(0) - \frac{128}{4275}(0.0284) - \frac{2197}{75240}(0.0990141) + \frac{1}{50}(0.116032) \\ &+ \frac{2}{55}(0.0397700)\end{aligned}\right) \\ &= 0.0001011 \end{aligned}$$

$$\begin{aligned} y(t_1) &= \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t} \\ &= \frac{1}{5}(0.2093900)e^{0.62817} - \frac{1}{25}e^{0.62817} + \frac{1}{25}e^{-0.41878} \\ &= 0.0298337 \end{aligned}$$

Since

$$R \le 0.0001$$

The value 0.0298184 0f approximation can be accepted

Proceeding in the same way to obtain the following table

i	t_i	w_i	h_i	y_i
1	0.2093900	0.0298184	0.2093900	0.0298337
3	0.5610469	0.4016438	0.1777496	0.4016860

Comment

Step 2 of 6

5

0.8387744

1.5894061

0.1280905

1.58934600

7

1.0000000

3.2190497

0.0486737

3.2190993

Comment

Step 3 of 6

The initial conditions give

$$t_0 = 2$$
$$w_0 = 1$$

To determine w_1 using $h = 0.25$, the maximum allowable step size,

compute $k_1, k_2, k_3, k_4, k_5, k_6$

$$\begin{aligned} k_1 &= hf(t_0, w_0) \\ &= 0.25\left(1 + (2-1)^2\right) \\ &= 0.25(1+1) \\ &= 0.5 \end{aligned}$$

$$\begin{aligned} k_2 &= hf\left(t_0 + \frac{1}{4}h, w_0 + \frac{1}{4}k_1\right) \\ &= 0.25 f\left(2 + \frac{1}{4}(0.25), 1 + \frac{1}{4}(0.5)\right) \\ &= 0.25\left(1 + (2.0625 - 1.125)^2\right) \\ &= 0.4697266 \end{aligned}$$

$$\begin{aligned} k_3 &= hf\left(t_0 + \frac{3}{8}h, w_0 + \frac{3}{32}k_1 + \frac{9}{32}k_2\right) \\ &= 0.25 f\left(2 + \frac{3}{8}(0.25), 1 + \frac{3}{32}(0.5) + \frac{9}{32}(0.4697266)\right) \\ &= 0.25 f(2.09375, 1.17899) \\ &= 0.25\left(1 + (2.09375 - 1.17899)^2\right) \\ &= 0.4591965 \end{aligned}$$

$$\begin{aligned} k_4 &= hf\left(t_0 + \frac{12}{14}h, w_0 + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3\right) \\ &= 0.25 f\left(2 + \frac{12}{13}(0.25), 1 + \frac{1932}{2197}(0.5) - \frac{7200}{2197}(0.4697266) + \frac{7296}{2197}(0.4592)\right) \\ &= 0.25 f(2.23077, 1.42526) \\ &= 0.25\left(1 + (2.23077 - 1.42526)^2\right) \\ &= 0.4122196 \end{aligned}$$

$$\begin{aligned} k_5 &= hf\left(t_0 + h, w_0 + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4\right) \\ &= 0.25 f\left(\begin{aligned}&2 + 0.25, 1 + \frac{439}{216}(0.5) - 8(0.4697266) + \frac{3680}{513}(0.4591965) \\ &- \frac{845}{4104}(0.4122116)\end{aligned}\right) \\ &= 0.25 f(2.25, 1.46756) \\ &= 0.25\left(1 + (2.25 - 1.46756)^2\right) \\ &= 0.4030531 \end{aligned}$$

$$\begin{aligned} k_6 &= hf\left(t_0 + \frac{1}{2}h, w_0 - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5\right) \\ &= 0.25 f\left(\begin{aligned}&2 + \frac{1}{2}(0.25), 1 - \frac{8}{27}(0.5) + 2(0.4697266) - \frac{3544}{2565}(0.4591965) \\ &+ \frac{1859}{4104}(0.4122116) - \frac{11}{40}(0.4030531)\end{aligned}\right) = 0.4490364 \\ &= 0.25 f(2.125, 1.23273) \\ &= 0.25\left(1 + (2.125 - 1.23273)^2\right) \end{aligned}$$

$$\begin{aligned} w_1 &= w_0 + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6 \\ &= 1 + \frac{16}{135}(0.5) + \frac{6656}{12825}(0.4591965) + \frac{28561}{56430}(0.4122116) - \frac{9}{50}(0.4030531) \\ &\quad + \frac{2}{55}(0.4490364) \\ &= 1.4499988 \end{aligned}$$

This implies that

$$\begin{aligned} R &= \frac{1}{h}\left|\frac{1}{360}k_1 - \frac{128}{4275}k_3 - \frac{2197}{75240}k_4 + \frac{1}{50}k_5 + \frac{2}{55}k_6\right| \\ &= \frac{1}{0.25}\left|\begin{aligned}&\frac{1}{360}(0.5) - \frac{128}{4275}(0.4591965) - \frac{2197}{75240}(0.4122116) + \frac{1}{50}(0.4030531) \\ &+ \frac{2}{55}(0.4490364)\end{aligned}\right| \\ &= 0.00002812 \end{aligned}$$

Let ε be tolerace

$$\begin{aligned} \text{Let } q &= 0.84\left(\frac{\varepsilon}{R}\right)^{\frac{1}{4}} \\ &= 0.84\left(\frac{0.0001}{0.00002812}\right) \\ &= 2.9871977 \end{aligned}$$

Comment

Step 4 of 6

Since

$$R \le 10^{-4}$$

the approximation can be taken as 1.4499988.

$$t_1 = t_0 + h$$

$$\begin{aligned} y(t_1) &= t_1 + \frac{1}{1 - t_1} \\ &= 2.25 + \frac{1}{1 - 2.25} \\ &= 2.25 - \frac{1}{1.25} \\ &= 1.45 \end{aligned}$$

$$\begin{aligned} k_1 &= hf(t_1, w_1) \\ &= 0.25\left(1 + (2.25 - 1.4499988)^2\right) \\ &= 0.4100005 \end{aligned}$$

$$\begin{aligned} k_2 &= hf\left(t_1 + \frac{1}{4}h, w_1 + \frac{1}{4}k_1\right) \\ &= 0.25 f\left(2.25 + \frac{1}{4}(0.25), 1.4499988 + \frac{1}{4}(0.4100005)\right) \\ &= 0.25 f(2.3125, 1.5525) \\ &= 0.25\left(1 + (2.3125 - 1.5525)^2\right) \\ &= 0.3944 \end{aligned}$$

$$\begin{aligned} k_3 &= hf\left(t_1 + \frac{3}{8}h, w_1 + \frac{3}{32}k_1 + \frac{9}{32}k_2\right) \\ &= 0.25 f\left(2.25 + \frac{3}{8}(0.25), 1.4499988 + \frac{3}{32}(0.4100005) + \frac{9}{32}(0.3944)\right) \\ &= 0.25 f(2.34375.1.59936) \\ &= 0.25\left(1 + (2.34375 - 1.59936)^2\right) \\ &= 0.3885291 \end{aligned}$$

$$\begin{aligned} k_4 &= hf\left(t_1 + \frac{12}{13}h, w_1 + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3\right) \\ &= 0.25 f\left(\begin{aligned}&2.25 + \frac{12}{13}(0.25), 1.4499988 + \frac{1932}{2197}(0.4100005) - \frac{7200}{2197}(0.3944) \\ &+ \frac{7296}{2197}(0.3885291)\end{aligned}\right) \\ &= 0.25 f(2.48077, 1.80828) \\ &= 0.25\left(1 + (2.48077 - 1.80828)^2\right) \\ &= 0.3630607 \end{aligned}$$

$$\begin{aligned} k_5 &= hf\left(t_1 + h, w_1 + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4\right) \\ &= 0.25 f\left(\begin{aligned}&2.25 + 0.25, 1.4499988 + \frac{439}{216}(0.4100005) - 8(0.3944) + \frac{3680}{513}(0.3885291) \\ &- \frac{845}{4104}(0.3630607)\end{aligned}\right) \\ &= 0.25 f(2.5, 1.84044) \\ &= 0.25\left(1 + (2.5 - 1.84044)^2\right) \\ &= 0.3587548 \end{aligned}$$

$$\begin{aligned} k_6 &= hf\left(t_1 + \frac{h}{2}, w_1 - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5\right) \\ &= 0.25 f\left(\begin{aligned}&2.25 + \frac{0.25}{2}, 1.449998 - \frac{8}{27}(0.4100005) + 2(0.3944) - \frac{3544}{2565}(0.3885291) \\ &+ \frac{1859}{4104}(0.3630607) - \frac{11}{40}(0.3587548)\end{aligned}\right) \\ &= 0.25 f(2.375, 1.64629) \\ &= 0.25\left(1 + (2.375 - 1.64629)^2\right) \\ &= 0.3827546 \end{aligned}$$

$$\begin{aligned} w_2 &= w_1 + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6 \\ &= 1.4499988 + \frac{6}{135}(0.4100005) + \frac{6656}{12825}(0.3885291) + \frac{28561}{56430}(0.3630607) \\ &\quad - \frac{9}{50}(0.3587548) + \frac{2}{55}(0.3827546) \\ &= 1.8333332 \end{aligned}$$

$$\begin{aligned} R &= \frac{1}{h}\left|\frac{1}{360}k_1 - \frac{128}{4275}k_3 - \frac{2197}{75240}k_4 + \frac{1}{50}k_5 + \frac{2}{55}k_6\right| \\ &= \frac{1}{0.25}\left|\begin{aligned}&\frac{1}{360}(0.4100005) - \frac{128}{4275}(0.3885291) - \frac{2197}{75240}(0.3630607) + \frac{1}{50}(0.3587548) \\ &+ \frac{2}{55}(0.3827546)\end{aligned}\right| \\ &= 0.0000086 \end{aligned}$$

$$\begin{aligned} q &= 0.84\left(\frac{\varepsilon}{R}\right)^{\frac{1}{4}} \\ &= 0.84\left(\frac{0.0001}{0.0000086}\right) \\ &= 9.7674419 \end{aligned}$$

Since,

$R \le 0.0001$ approximation can be taken as 1.8333332

Repeating the above steps to obtain the following table

i	t_i	w_i	h_i	y_i
1	2.2500000	1.4499988	0.2500000	1.4500000
2	2.5000000	1.8333332	0.2500000	1.8333333
3	2.7500000	2.1785718	0.2500000	2.1785714
4	3.0000000	2.5000005	0.2500000	2.5000000

Comment

Step 5 of 6

The initial conditions give

$$t_0 = 1$$
$$w_0 = 2$$

To determine

$$w_1$$

Using $h = 1.25$, the maximum allowable size.

Compute

$$\begin{aligned} k_1 &= hf(t_0, w_0) \\ &= 0.25 f(1, 2) \\ &= 0.25(1 + 2) \\ &= 0.75 \end{aligned}$$

$$\begin{aligned} k_2 &= hf\left(t_0 + \frac{1}{4}h, w_0 + \frac{1}{4}k_1\right) \\ &= 0.25 f\left(1 + \frac{1}{4}(0.25), 2 + \frac{1}{4}(0.75)\right) \\ &= 0.25 f(1.0625, 2.1875) \\ &= 0.25\left(1 + \frac{2.1875}{1.0625}\right) \\ &= 0.7647059 \end{aligned}$$

$$\begin{aligned} k_3 &= hf\left(t_0 + \frac{3}{8}h, w_0 + \frac{3}{32}k_1 + \frac{9}{32}k_2\right) \\ &= 0.25 f\left(1 + \frac{3}{8}(0.25), 2 + \frac{3}{32}(0.75) + \frac{9}{32}(0.7647059)\right) \\ &= 0.25 f(1.09375, 2.28539) \\ &= 0.25\left(1 + \frac{2.28539}{1.09375}\right) \\ &= 0.7723749 \end{aligned}$$

$$\begin{aligned} k_4 &= hf\left(t_0 + \frac{12}{13}h, w_0 + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3\right) \\ &= 0.25 f\left(\begin{aligned}&1 + \frac{12}{13}(0.25), 2 + \frac{1932}{2197}(0.75) - \frac{7200}{2197}(0.7647059) \\ &+ \frac{7296}{2197}(0.7723749)\end{aligned}\right) \\ &= 0.25 f(1.23077, 2.71842) \\ &= 0.25\left(1 + \frac{2.71842}{1.23077}\right) \\ &= 0.8021787 \end{aligned}$$

$$\begin{aligned} k_5 &= hf(t_0 + h, w_0 + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4 \\ &= 0.25 f\left(\begin{aligned}&1 + 0.25, 2 + \frac{439}{216}(0.75) - 8(0.7647059) + \frac{3680}{513}(0.7723749) \\ &- \frac{845}{4104}(0.8021787)\end{aligned}\right) \\ &= 0.25 f(1.25, 2.78212) \\ &= 0.25\left(1 + \frac{2.78212}{1.25}\right) \\ &= 0.806424 \end{aligned}$$

$$\begin{aligned} k_6 &= hf\left(t_0 + \frac{h}{2}, w_0 - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5\right) \\ &= 0.25 f\left(\begin{aligned}&1 + \frac{0.25}{2}, 2 - \frac{8}{27}(0.75) + 2(0.7647059) - \frac{3544}{2565}(0.7723749) \\ &+ \frac{1859}{4104}(0.8021787) - \frac{11}{40}(0.806424)\end{aligned}\right) \\ &= 0.25 f(1.125, 2.38162) \\ &= 0.25\left(1 + \frac{2.38162}{1.125}\right) \\ &= 0.7792489 \end{aligned}$$

$$\begin{aligned} w_1 &= w_0 + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6 \\ &= 2 + \frac{6}{135}(0.75) + \frac{6656}{12825}(0.7723749) + \frac{28561}{56430}(0.8021787) - \frac{9}{50}(0.806424) \\ &\quad + \frac{2}{55}(0.7792489) \\ &= 2.7789299 \end{aligned}$$

$$\begin{aligned} R &= \frac{1}{h}\left|\frac{1}{360}k_1 - \frac{128}{4275}k_3 - \frac{2197}{75240}k_4 + \frac{1}{50}k_5 + \frac{2}{55}k_6\right| \\ &= \frac{1}{0.25}\left|\begin{aligned}&\frac{1}{360}(0.75) - \frac{128}{4275}(0.7723749) - \frac{2197}{75240}(0.8021787) + \frac{1}{50}(0.806424) \\ &+ \frac{2}{55}(0.7792489)\end{aligned}\right| \\ &= 0.0000059 \end{aligned}$$

$$\begin{aligned} q &= 0.84\left(\frac{0.0001}{0.0000059}\right)^{\frac{1}{4}} \\ &= 1.70438 \end{aligned}$$

Since $R \le 0.0001$, the approximation is 2.7789299.

Now

$$w_1 = 2.7789299$$
$$t_1 = 1.25$$

$$\begin{aligned} k_1 &= hf(t_1, w_1) \\ &= 0.25 f(1.25, 2.7789299) \\ &= 0.25\left(1 + \frac{2.7789299}{1.25}\right) \\ &= 0.805786 \end{aligned}$$

$$= 0.25f\left(1.25+\frac{1}{4}(0.25), 2.7789299+\frac{1}{4}(0.805786)\right)$$
$$= 0.25f(1.3125, 2.98038)$$
$$= 0.25\left(1+\frac{2.98038}{1.3125}\right)$$
$$= 0.8176914$$

$$k_3 = hf\left(t_1+\frac{3}{8}h, w_1+\frac{3}{32}k_1+\frac{9}{32}k_2\right)$$
$$= 0.25f\left(1.25+\frac{3}{8}(0.25), 2.7789299+\frac{3}{32}(0.805786)+\frac{9}{32}(0.8176914)\right)$$
$$= 0.25f(1.34375, 3.08445)$$
$$= 0.25\left(1+\frac{3.08445}{1.34375}\right)$$
$$= 0.8238511$$

$$k_4 = hf\left(t_1+\frac{12}{13}h, w_1+\frac{1932}{2197}k_1-\frac{7200}{2197}k_2+\frac{7296}{2197}k_3\right)$$
$$= 0.25f\begin{pmatrix}1.25+\frac{12}{13}(0.25), 2.7789299+\frac{1932}{2197}(0.805786)-\frac{7200}{2197}(0.8176914)\\ +\frac{7296}{2197}(0.8238511)\end{pmatrix}$$
$$= 0.25f(1.48077, 3.54371)$$
$$= 0.25\left(1+\frac{3.54371}{1.48077}\right)$$
$$= 0.8482884$$

$$k_5 = hf\left(t_1+h, w_1+\frac{439}{216}k_1-8k_2+\frac{3680}{513}k_3-\frac{845}{4104}k_4\right)$$
$$= 0.25f\begin{pmatrix}1.25+0.25, 2.7789299+\frac{439}{216}(0.805786)-8(0.8176914)\\ +\frac{3680}{513}(0.8238511)\\ -\frac{845}{4104}(0.8482884)\end{pmatrix}$$
$$= 0.25f(1.5, 3.61031)$$
$$= 0.25\left(1+\frac{3.61031}{1.5}\right)$$
$$= 0.8517183$$

$$k_6 = hf\left(t_1+\frac{h}{2}, w_1-\frac{8}{27}k_1+2k_2-\frac{3544}{2565}k_3+\frac{1859}{4104}k_4-\frac{11}{40}k_5\right)$$
$$= 0.25f\begin{pmatrix}1.25+\frac{0.25}{2}, 2.7789299-\frac{8}{27}(0.805786)+2(0.8176914)\\ -\frac{3544}{2565}(0.8238511)\\ +\frac{1859}{4104}(0.8482884)-\frac{11}{40}(0.8517183)\end{pmatrix}$$
$$= 0.25f(1.375, 3.18729)$$
$$= 0.25\left(1+\frac{3.18729}{1.375}\right)$$
$$= 0.8905073$$

$$w_2 = w_1+\frac{16}{135}k_1+\frac{6656}{12825}k_3+\frac{28561}{56430}k_4-\frac{9}{50}k_5+\frac{2}{55}k_6$$
$$= 2.7789299+\frac{16}{135}(0.805786)+\frac{6656}{12825}(0.8238511)+\frac{28561}{56430}(0.8482884)$$
$$-\frac{9}{50}(0.8517183)+\frac{2}{55}(0.8995073)$$
$$= 3.6081985$$

$$R = \frac{1}{h}\left|\frac{1}{360}k_1-\frac{128}{1275}k_3-\frac{2197}{75240}k_4+\frac{1}{50}k_5+\frac{2}{55}k_6\right|$$
$$= \frac{1}{0.25}\left|\begin{matrix}\frac{1}{369}(0.805786)-\frac{128}{1275}(0.8238511)-\frac{2197}{75240}(0.8482884)\\ +\frac{1}{50}(0.8517183)\\ +\frac{2}{55}(0.8995073)\end{matrix}\right|$$
$$= 0.2222028$$

$$q = 0.84\left(\frac{\varepsilon}{R}\right)^{\frac{1}{4}}$$
$$= 0.84\left(\frac{0.0001}{0.2222028}\right)^{\frac{1}{4}}$$
$$= 0.122347$$

Since

$R \le 0.0001$

The approximation is 3.6081985

Proceed as above to obtain the following table

i	t_i	w_i	h_i	y_i
1	1.2500000	2.7789299	0.2500000	2.7789294
2	1.5000000	3.6081985	0.2500000	3.6081977
3	1.7500000	4.4793288	0.2500000	4.4793276
4	2.0000000	5.3862958	0.2500000	5.3862944

The initial conditions give

$t_0 = 0$
$w_0 = 1$

To determine w_1 using $h = 0.25$, the maximum allowable step size

Compute

$$k_1 = hf(t_0, w_0)$$
$$= 0.25f(0,1)$$
$$= 0.25(\cos 0+\sin 0)$$
$$= 0.25$$

$$k_2 = hf\left(t_0+\frac{1}{4}h, w_0+\frac{1}{4}k_1\right)$$
$$= 0.25f\left(0+\frac{1}{4}(0.25), 1+\frac{1}{4}(0.25)\right)$$
$$= 0.25f(0.0625.1.0625)$$
$$= 0.25(\cos(0.125)+\sin(0.1875))$$
$$= 0.294650$$

$$k_3 = hf\left(t_0+\frac{3}{8}h, w_0+\frac{3}{32}k_1+\frac{9}{32}k_2\right)$$
$$= 0.25f\left(0+\frac{3}{8}(0.25), 1+\frac{3}{32}(0.25)+\frac{9}{32}(0.294650)\right)$$
$$= 0.25f(0.09375, 1.10631)$$
$$= 0.25(\cos(0.1875)+\sin(0.28125))$$

$$k_4 = hf\left(t_0+\frac{12}{13}h, w_0+\frac{1932}{2197}k_1-\frac{7200}{2197}k_2+\frac{7296}{2197}k_3\right)$$
$$= 0.25f\left(0+\frac{12}{13}(0.25), 1+\frac{1932}{2197}(0.25)-\frac{7200}{2197}(0.294650)+\frac{7296}{2197}(0.315008)\right)$$
$$= 0.25f(0.230769, 1.30033)$$
$$= 0.25(\cos(0.461538)+\sin(0.692307))$$
$$= 0.383421$$

$$k_5 = hf\left(t_0+h, w_0+\frac{439}{216}k_1-8k_2+\frac{3680}{513}k_3-\frac{845}{513}k_4\right)$$
$$= 0.25f\left(0+0.25, 1+\frac{439}{216}(0.25)-8(0.294650)+\frac{3680}{513}(0.315008)-\frac{845}{513}(0.383421)\right)$$
$$= 0.25f(0.25, 0.779047)$$
$$= 0.25(\cos(0.5)+\sin(0.75))$$
$$= 0.389805$$

$$k_6 = hf\left(t_0+\frac{h}{2}, w_0-\frac{8}{27}k_1+2k_2-\frac{3544}{2565}k_3+\frac{1859}{4104}k_4-\frac{11}{40}k_5\right)$$
$$= 0.25f\left(0+\frac{0.25}{2}, 1-\frac{8}{27}(0.25)+2(0.294650)-\frac{3544}{2565}(0.315008)+\frac{1859}{4104}(0.383421)-\frac{11}{40}(0.389805)\right)$$
$$= 0.25f(0.125, 1.14647)$$
$$= 0.25(\cos(0.25)+\sin(0.375))$$
$$= 0.333796$$

$$w_1 = w_0+\frac{16}{135}k_1+\frac{6656}{12825}k_3+\frac{28561}{56430}k_4-\frac{9}{50}k_5+\frac{2}{55}k_6$$
$$= 1+\frac{16}{135}(0.25)+\frac{6656}{12825}(0.315008)+\frac{28561}{56430}(0.383421)-\frac{9}{50}(0.389805)+\frac{2}{55}(0.333796)$$
$$= 1.3291478$$

$$R = \frac{1}{h}\left|\frac{1}{360}k_1-\frac{128}{4275}k_3-\frac{2197}{75240}k_4+\frac{1}{50}k_5+\frac{2}{55}k_6\right|$$
$$= \frac{1}{0.25}\left|\frac{1}{360}(0.25)-\frac{128}{4275}(0.315008)-\frac{2197}{75240}(0.383421)+\frac{1}{50}(0.389805)+\frac{2}{55}(0.333796)\right|$$
$$= 0.0000036$$

$$q = 0.84\left(\frac{\varepsilon}{R}\right)^{\frac{1}{4}}$$
$$= 0.84\left(\frac{0.0001}{0.0000036}\right)$$
$$= 23.3333333$$

Since $R \le 0.0001$

The value 1.3291478 of approximation is accepted.

$$y(t_1) = \frac{1}{2}\sin 2t_1-\frac{1}{3}\cos 3t_1+\frac{4}{3}$$
$$= \frac{1}{2}\sin(0.5)-\frac{1}{3}\cos(0.75)+\frac{4}{3}$$
$$= 1.3291498$$

Now to find w_2

$$k_1 = hf(t_1, w_1)$$
$$= 0.25f(0.25, 1.3291478)$$
$$= 0.25(\cos(0.5)+\sin(0.75))$$
$$= 0.389805$$

Comment

Step 4 of 5

$$k_2 = hf\left(t_1+\frac{1}{4}h, w_1+\frac{1}{4}k_1\right)$$
$$= 0.25f\left(0.25+\frac{1}{4}(0.25), 1.3291478+\frac{1}{4}(0.389805)\right)$$
$$= 0.25f(0.3125, 1.426599)$$
$$= 0.25(\cos(0.625)+\sin(0.937500))$$
$$= 0.404261$$

$$k_3 = hf\left(t_1+\frac{3}{8}h, w_1+\frac{3}{32}k_1+\frac{9}{32}k_2\right)$$
$$= 0.25f\left(0.25+\frac{3}{8}(0.25), 1.3291478+\frac{3}{32}(0.389805)+\frac{9}{32}(0.404261)\right)$$
$$= 0.25f(0.34375, 1.47939)$$
$$= 0.25(\cos(0.6875)+\sin(1.03125))$$
$$= 0.407694$$

$$k_4 = hf\left(t_1+\frac{12}{13}h, w_1+\frac{1932}{2197}k_1-\frac{7200}{2197}k_2+\frac{7296}{2197}k_3\right)$$
$$= 0.25f\left(0.25+\frac{12}{13}(0.25), 1.3291478+\frac{1932}{2197}(0.389805)-\frac{7200}{2197}(0.404261)+\frac{7296}{2197}(0.407694)\right)$$
$$= 0.25f(0.480769, 1.701)$$
$$= 0.25(\cos(0.961538)+\sin(1.442307))$$
$$= 0.391004$$

$$k_5 = hf\left(t_1+h, w_1+\frac{439}{216}k_1-8k_2+\frac{3680}{513}k_3-\frac{845}{4104}k_4\right)$$
$$= 0.25f\left(0.25+0.25, 1.3291478+\frac{439}{216}(0.389805)-8(0.404261)+\frac{3680}{513}(0.407694)-\frac{845}{4104}(0.391004)\right)$$
$$= 0.25f(0.5, 1.73138)$$
$$= 0.25(\cos(1)+\sin(1.5))$$
$$= 0.384449$$

$$k_6 = hf(t_1+\frac{1}{2}h, w_1-\frac{8}{27}k_1+2k_2-\frac{3544}{2565}k_3+\frac{1859}{4104}k_4-\frac{11}{40}k_5$$
$$= 0.25f(0.375, 1.53026)$$
\
$$= 0.25(\cos(0.75)+\sin(1.125))$$
$$= 0.408489$$

$$w_2 = w_1+\frac{16}{135}k_1+\frac{6656}{12825}k_3+\frac{28561}{56430}k_4-\frac{9}{50}k_5+\frac{2}{55}k_6$$
$$= 1.3291478+\frac{16}{135}(0.389805)+\frac{6656}{12825}(0.407694)+\frac{28561}{56430}(0.391004)-\frac{9}{50}(0.384449)+\frac{2}{55}(0.408489)$$
$$= 1.7304857$$

$$R = \frac{1}{h}\left|\frac{1}{360}k_1-\frac{128}{4275}k_3-\frac{2197}{75240}k_4+\frac{1}{50}k_5+\frac{2}{55}k_6\right|$$
$$= \frac{1}{0.25}\left|\frac{1}{360}(0.389805)-\frac{128}{4275}(0.407694)-\frac{2197}{75240}(0.391004)+\frac{1}{50}(0.384449)+\frac{2}{55}(0.408489)\right|$$
$$= 0.0000066$$

$$q = 0.84\left(\frac{\varepsilon}{R}\right)^{\frac{1}{4}}$$
$$= 0.84\left(\frac{0.0001}{0.000006}\right)^{\frac{1}{4}}$$
$$= 1.69723$$

Since $R \le 0.0001$ the approximation is 1.7304857.

Proceeding as above, we get the following table

i	t_i	w_i	h_i	y_i
1	0.2500000	1.3291478	0.2500000	1.3291498
2	0.5000000	1.7304857	0.2500000	1.7304898
3	0.7500000	2.0414669	0.2500000	2.0414720
4	1.0000000	2.1179750	0.2500000	2.1179795

Comment

Was this solution helpful? 1 0

Recommended solutions for you in Chapter 5.5